FAILURE OF TAMENESS FOR LOCAL COHOMOLOGY

STEVEN DALE CUTKOSKY AND JÜRGEN HERZOG

Abstract. We give an example that shows that not all local cohomology modules are tame in the sense of Brodmann and Hellus.

Introduction

In their paper "Cohomological patterns of coherent sheaves over projective schemes", Brodmann and Hellus [BrHe] raised the following tameness problem: let $R = \bigoplus_{n\geq 0} R_n$ be a positively graded Noetherian ring such that R_0 is semilocal, and let M be a finitely generated graded R-module. Denote by J the graded ideal $\bigoplus_{n>0} R_n$. Is true that all the local cohomology modules $H^i_J(M)$ are tame? The authors call a graded R-module T tame, if there exists an integer n_0 such that $T_n = 0$ for all $n \leq n_0$, or else $T_n \neq 0$ for all $n \leq n_0$.

The tameness problem has been answered in the affirmative in many cases, in particular, if $\dim R_0 \leq 2$. We refer to the article [Br] of Brodmann for a survey on this problem. In this paper we present an example that shows that the tameness problem does not always have a positive answer. In our example, R_0 is a 3-dimensional normal local ring with isolated singularity.

1. A bigraded ring with periodic local cohomology

Suppose that $\mathcal{A}$ is a very ample line bundle on a projective space $\mathbb{P}^n$, and $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^n$. $\mathcal{F}$ is m-regular with respect to $\mathcal{A}$ if $H^i(\mathbb{P}^n, \mathcal{F} \otimes \mathcal{A}^{\otimes(m-i)}) = 0$ for all $i > 0$.

If $\mathcal{F}$ is m-regular with respect to $\mathcal{A}$, then $H^0(\mathbb{P}^n, \mathcal{F} \otimes \mathcal{A}^{\otimes k})$ is spanned by

$$H^0(\mathbb{P}^n, \mathcal{F} \otimes \mathcal{A}^{\otimes(k-1)}) \otimes H^0(\mathbb{P}^n, \mathcal{A})$$

if $k > m$ (Lecture 14 [Mu]).

Theorem 1.1. *Suppose that* $\mathbf{k}$ *is an algebraically closed field. Then there exists a bigraded domain*

$$R = \sum_{m,n\geq 0} R_{m,n} t^m u^n$$

with the following properties:

(1) *R is of finite type over $R_{0,0} = \mathbf{k}$, and is generated in degree 1 over $R_{0,0}$ (with respect to the grading $d(f) = m + n$ for $f \in R_{m,n} t^m u^n$).*
(2) *R has dimension 4, is normal, and the singular locus of* $\operatorname{Spec}(R)$ *is the bigraded maximal ideal $\sum_{m+n>0} R_{m,n} t^m u^n$.*

The first author was partially supported by NSF.

(3) *For $n \geq 0$, let*

$$R_n = \sum_{m \geq 0} R_{m,n} t^m.$$

R_0 is a normal, 3 dimensional graded ring, the R_n are graded R_0 modules, and $R = \sum_{n \geq 0} R_n u^n$ is generated by R_1 as an R_0 algebra.

(4) *Let $I = \sum_{m>0} R_{m,0} t^m$ be the graded maximal ideal of R_0. The singular locus of $\mathrm{Spec}(R_0)$ is I.*

(5) *For $n \geq 0$, we have*

$$H_I^2(R_n) = \begin{cases} \mathbf{k}^2 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let E be an elliptic curve over $\mathbf{k}$, and $\overline{D}$ be a degree 0 divisor on E such that $\overline{D}$ has order 2 in the Jacobian of E ($\overline{D} \not\sim 0$ and $2\overline{D} \sim 0$). Let $p \in E$ be a (closed) point. Let $S = E \times_{\mathbf{k}} E$, with projections $\pi_1 : S \to E$ and $\pi_2 : S \to E$. Let $H = \pi_1^*(p) + \pi_2^*(p)$ and $D = \pi_2^*(\overline{D})$ be divisors on S. H and $D + H$ are ample on S by the Nakai criterion (Theorem 5.1 [Ha]).

Suppose that $m, n \in \mathbf{Z}$.

$$\begin{aligned} H^1(S, \mathcal{O}_S(mH + nD)) \cong\ & H^0(E, \mathcal{O}_E(mp)) \otimes_{\mathbf{k}} H^1(E, \mathcal{O}_E(mp + n\overline{D})) \\ & \oplus H^1(E, \mathcal{O}_E(mp)) \otimes_{\mathbf{k}} H^0(E, \mathcal{O}_E(mp + n\overline{D})) \end{aligned}$$

by the Künneth formula (IV of Lecture 11 [Mu]). If $m < 0$, then

$$H^0(E, \mathcal{O}_E(mp)) = 0 \text{ and } H^0(E, \mathcal{O}_E(mp + n\overline{D})) = 0.$$

If $m > 0$, then by Serre duality,

$$H^1(E, \mathcal{O}_E(mp)) \cong H^0(E, \mathcal{O}_E(-mp)) = 0$$

and

$$H^1(E, \mathcal{O}_E(mp + n\overline{D})) \cong H^0(E, \mathcal{O}_E(-mp - n\overline{D})) = 0.$$

If $m = 0$, we have

$$H^1(S, \mathcal{O}_S(nD)) \cong H^1(E, \mathcal{O}_E(n\overline{D})) \oplus H^0(E, \mathcal{O}_E(n\overline{D})).$$

By the Riemann-Roch theorem on E, we have for $m, n \in \mathbf{Z}$,

$$h^1(S, \mathcal{O}_S(mH + nD)) = \begin{cases} 2 & \text{if } m = 0 \text{ and } n \text{ is even} \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Let $\mathcal{E} = \mathcal{O}_S \oplus \mathcal{O}_S(D)$. Let $X = \mathbf{P}(\mathcal{E})$ be the projective space bundle ([Ha, Section II.7]) with projection $\pi : X \to S$ and associated line bundle $\mathcal{O}_X(1)$. Since H is ample on S, there exists a number τ such that $\pi^*\mathcal{O}_S(nH) \otimes_{\mathcal{O}_X} \mathcal{O}_X(1)$ is very ample on X for all $n \geq \tau$ ([Ha, Proposition II.7.10 and Exercise II.5.12(b)]).

There exists $l \geq \tau$ such that lH and $D + lH \sim l(D + H)$ are very ample.

There exists an odd number $r_2 > 0$ such that $H^i(S, \mathcal{O}_S((r_2 - i)(D + alH)) = 0$ for all $a \geq 1$ and $i > 0$. Thus for all $a \geq 1$, $\mathcal{O}_S$ is r_2 regular for $D + alH$. It follows that $H^0(S, \mathcal{O}_S(mr_2(D + alH))$ is spanned by $H^0(S, \mathcal{O}_S(r_2(D + alH)))^{\otimes m}$ for all $a \geq 1$ and $m \geq 1$.

There exists $r_1 > 0$ such that $\mathcal{O}_S$ and $\mathcal{O}_S(D)$ are r_1 regular for lH. Thus $\mathcal{O}_S(nD)$ is r_1 regular for lH for all $n \in \mathbb{Z}$ (since $2D \sim 0$).

Choose $a > r_1$. For $m, n \geq 0$, $H^0(S, \mathcal{O}_S(mr_2(D+alH)+nr_2alH))$ is spanned by $H^0(S, \mathcal{O}_S(mr_2(D+alH)) \otimes H^0(S, \mathcal{O}_S(r_2alH))^{\otimes n}$ since $\mathcal{O}_S(mr_2D)$ is r_1 regular for lH.

Thus $H^0(S, \mathcal{O}_S(mr_2(D+alH)+nr_2alH))$ is spanned by

$$H^0(S, \mathcal{O}_S(r_2(D+alH))^{\otimes m} \otimes H^0(S, \mathcal{O}_S(r_2alH))^{\otimes n}.$$

Let

$$R_{m,n} = H^0(S, \mathcal{O}_S(mr_2laH + nr_2(D+alH))).$$

Let

$$R_n = \sum_{m \geq 0} R_{m,n} t^m,$$

$$R = \sum_{m,n \geq 0} R_{m,n} t^m u^n = \sum_{n \geq 0} R_n u^n,$$

where t, u are indeterminates. R is normal (by [Z, Theorem 4.2]). By our construction, R is generated as an $R_{0,0} = \mathbf{k}$ algebra by $R_{1,0}t$ and $R_{0,1}u$. We deduce that R is standard graded as an R_0 algebra, R_0 is normal and standard graded as an $R_{0,0} = \mathbf{k}$ algebra. We have an isomorphism $S \cong \mathrm{Proj}(R_0)$, with associated line bundle $\mathcal{O}_S(1) \cong \mathcal{O}_S(r_2alH)$. Since S is nonsingular, the singular locus of $\mathrm{Spec}(R_0)$ is the graded maximal ideal $I = \sum_{m>0} R_{m,0}t^m$ of R_0. The sheafication of the module R_n on S is

$$\tilde{R}_n = \mathcal{O}_S(nr_2(D+alH)),$$

by Exercise II.5.9 (c) [Ha]. For $n \geq 0$, by (1), and the exact sequences relating local cohomology and global cohomology ([E, A.4.1]), we have

$$H^2_I(R_n) \cong \oplus_{m \in \mathbf{Z}} H^1(S, \mathcal{O}_S(mr_2alH + nr_2(D+alH))) \cong H^1(S, \mathcal{O}_S(nr_2D)).$$

Thus

$$H^2_I(R_n) \cong \begin{cases} \mathbf{k}^2 & \text{if } n \text{ is even}, \\ 0 & \text{if } n \text{ is odd}. \end{cases} \tag{2}$$

Let $\mathcal{L} = \pi^*\mathcal{O}_S(r_2alH) \otimes_{\mathcal{O}_X} \mathcal{O}_X(1)$. By our choice of l, $\mathcal{L}$ is very ample on X. For $r \geq 0$, we have (by [Ha, Proposition II.7.11])

$$\begin{aligned} H^0(X, \mathcal{L}^r) &\cong H^0(S, \mathrm{Sym}^r(\mathcal{E}) \otimes_{\mathcal{O}_S} \mathcal{O}_S(rr_2alH)) \\ &\cong \oplus_{n=0}^r H^0(S, \mathcal{O}_S(rr_2alH + nD)) \\ &\cong \oplus_{n=0}^r H^0(S, \mathcal{O}_S(rr_2alH + nr_2D)) \text{ (since } r_2 \text{ is odd and } D \text{ has order 2)} \\ &\cong \oplus_{n=0}^r H^0(S, \mathcal{O}_S((r-n)r_2alH + nr_2(D+alH))) \\ &\cong \textstyle\sum_{m+n=r} R_{m,n} t^m u^n. \end{aligned}$$

Thus R, with the above grading, is the coordinate ring of X, with respect to an embedding in projective space given by $H^0(X, \mathcal{L})$. Since X is nonsingular, the singular locus of $\mathrm{Spec}(R)$ is the bigraded maximal ideal $\sum_{m+n>0} R_{m,n} t^m u^n$ of R. $\square$

Choose a surjective homomorphism $S_0 \to R_0$ of graded $\mathbf{k}$-algebras, where S_0 is a polynomial ring of dimension d. Then the graded version of the local duality theorem ([BH, Theorem 3.6.19]) and property (5) of R imply that

$$\operatorname{Ext}_{S_0}^{d-2}(R_n, S_0) \cong \begin{cases} \mathbf{k}^2 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

More generally, if R is a positively graded Noetherian R_0-algebra, M is a finitely generated graded R-module, and N an R_0-module, one could ask whether the graded R-modules $\operatorname{Ext}^i_{R_0}(M,N)$, $\operatorname{Ext}^i_{R_0}(N,M)$ and $\operatorname{Tor}^i_{R_0}(M,N)$ behave tamely. The above example shows that this is in general not the case for $\operatorname{Ext}^i_{R_0}(M,N)$, while for the other two functors this is the case. In fact, computing $\operatorname{Ext}^i_{R_0}(N,M)$ and $\operatorname{Tor}^i_{R_0}(M,N)$ by using a graded minimal free R_0-resolution of N, these two homology groups are subquotients of a complex whose chains are a finite number of copies of M. Hence $\operatorname{Ext}^i_{R_0}(N,M)$ and $\operatorname{Tor}^i_{R_0}(M,N)$ are finitely generated graded R-modules. Say, n_0 is the highest degree of a generator of $\operatorname{Ext}^i_{R_0}(N,M)$. Then one has

$$\operatorname{Ext}^i_{R_0}(N,M_n)=0 \quad \text{for all } n \geq n_0, \qquad \text{or else} \qquad \operatorname{Ext}^i_{R_0}(N,M_n)\neq 0 \quad \text{for all } n \geq n_0.$$

The same holds true for $\operatorname{Tor}^i_{R_0}(M,N)$.

2. An example of a non-tame cohomology module

In this section we use the result of the previous section to produce an example which gives a negative answer to the tameness problem [BrHe, Problem 5.1] of Brodmann and Hellus. The construction is based on a duality theorem for bigraded modules which is given in [HR].

Let $\mathbf{k}$ be a field, $S=\mathbf{k}[x_1,\ldots,x_r,y_1,\ldots,y_s]$ the standard bigraded polynomial ring. In other words, we set $\deg x_i=(1,0)$ and $\deg y_j=(0,1)$ for all i,j. We denote by $P_0=(x_1,\ldots,x_r)$ the graded maximal ideal of $\mathbf{k}[x_1,\ldots,x_r]$ and by $Q_0=(y_1,\ldots,y_s)$ the graded maximal ideal of $\mathbf{k}[y_1,\ldots,y_s]$, and set $P=P_0S$, $Q=Q_0S$ and $S_+=P+Q$.

Let M be a finitely generated bigraded S-module. We set $M_j=\bigoplus_{i\in\mathbb{Z}} M_{(i,j)}$. Then $M=\bigoplus_{j\in\mathbb{Z}} M_j$, where each M_j is a finitely generated graded $\mathbf{k}[x_1,\ldots,x_r]$-module.

We denote by $M^\vee$ the bigraded $\mathbf{k}$-dual of M, i.e. the bigraded $\mathbf{k}$-module with components

$$(M^\vee)_{(i,j)}=\operatorname{Hom}_{\mathbf{k}}(M_{(-i,-j)},\mathbf{k}) \quad \text{for all} \quad i,j.$$

By the local duality theorem one has natural isomorphisms of bigraded S-modules

$$H^i_{S_+}(M)^\vee \cong \operatorname{Ext}^{r+s-i}(M,S(-r,-s)) \quad \text{for all} \quad i.$$

In particular, all the modules $H^i_{S_+}(M)^\vee$ are finitely generated bigraded S-modules, see [BH, Theorem 3.6.19] for a similar statement in the graded case.

We shall use the following result [HR, Proposition 2.5]

Proposition 2.1. *Suppose M is a finitely generated graded generalized Cohen-Macaulay S-module of dimension d (i.e. M is Cohen-Macaulay on the punctured spectrum of S, or equivalently, $H^i_{S_+}(M)$ has finite length for $i<d$). We let N be*

the finitely generated bigraded S-module $H^d_{S_+}(M)^\vee$. Then one has the following long exact sequence of bigraded S-modules

$$0 \to H^1_P(N) \to H^{d-1}_Q(M)^\vee \to H^{d-1}_{S_+}(M)^\vee \to H^2_P(N) \to H^{d-2}_Q(M)^\vee \to H^{d-2}_{S_+}(M)^\vee \cdots$$

Note that $(H^{d-i}_{S_+}(M)^\vee)_j = 0$ for $i > 0$ and all $j \gg 0$. Thus the long exact sequence of Proposition 2.1 yields the following isomorphisms

$$(H^{d-i}_Q(M)_{-j})^\vee \cong (H^{d-i}_Q(M)^\vee)_j \cong H^i_P(N)_j \cong H^i_{P_0}(N_j) \tag{3}$$

for all $i > 0$ and all $j \gg 0$.

Now let R be the bigraded $\mathbf{k}$-algebra of Theorem 1.1. We choose a bigraded presentation $S \to R$ with $S = \mathbf{k}[x_1, \ldots, x_r, y_1, \ldots, y_s]$, and view R a bigraded S-module.

We have $\dim R = 4$, so that $\omega_R = H^4_{S_+}(R)^\vee$ is the canonical module of R. Since R is a domain, the canonical module localizes, that is, we have $(\omega_R)_\wp \cong \omega_{R_\wp}$ for all $\wp \in \operatorname{Spec}(R)$, see for example [HK, Korollar 5.25]. Furthermore, since the singular locus of R is the bigraded maximal ideal $\mathfrak{m}$ of R, it follows that $R_\wp$ is regular for all $\wp \neq \mathfrak{m}$. In particular, $(\omega_R)_\wp \cong R_\wp$ is Cohen-Macaulay for all $\wp \neq \mathfrak{m}$. This shows that ω_R is a generalized Cohen-Macaulay module. Finally, since R is normal, R is in particular S_2, and this, by a result of Aoyama [A, Proposition 2], implies that $R \cong H^4_{S_+}(\omega_R)^\vee$. Thus if we set $J = QR = \bigoplus_{n>0} R_n$, then (3) applied to ω_R yields

$$(H^2_J(\omega_R)_{-j})^\vee \cong H^2_I(R_j) \quad \text{for} \quad j \gg 0.$$

Thus in view of property (5) of R we obtain

Corollary 2.2. *For all $j \ll 0$ one has*

$$H^2_J(\omega_R)_j \cong \begin{cases} \mathbf{k}^2 & \text{if } j \text{ is even} \\ 0 & \text{if } j \text{ is odd.} \end{cases}$$

Localizing at the graded maximal ideal of R_0 we may as well assume that R_0 is local and obtain the same result.

References

[A] Aoyama, *On the depth and the projective dimension of the canonical module*, Japan J. Math. **6**(1980), 61–69.

[BrHe] Brodmann, M. and Hellus, M., *Cohomological patterns of coherent sheaves over projective schemes*, J. Pure and Appl. Alg. 172 (2002), 165–182.

[Br] Brodmann, M., *Asymptotic behaviour of cohomology: tameness, supports and associated primes, Joint International Meeting of the American Mathematical Society and the Indian Mathematical Society on Commutative Algebra and Algebraic Geometry*, Bangalore/India, December 17-20, 2003, Contemporary Mathematics **390**(2005), 31-61.

[E] Eisenbud, D., *Commutative algebra, with a view towards algebraic geometry*, Springer Verlag, New York, Heidelberg, Berlin (1995).

[Ha] Hartshorne, R., *Algebraic Geometry*, Springer Verlag, New York, Heidelberg, Berlin, 1977.

[BH] Bruns, W. and Herzog, J., *Cohen-Macaulay rings (Revised edition)*, Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, 1998.

[HK] Herzog, J. and Kunz, E., *Der kanonische Modul eines Cohen-Macauly Rings*, Lecture Notes in Mathematics **238**, Springer, 1971.

[HR] Herzog, J. and Rahimi, A., *Local Duality for Bigraded Modules*, math.AC/0604587.

[L] Lim, C.S., *Tameness of graded local cohomology modules for dimension $R_0 = 2$, the Cohen-Macaulay case*, Menumi Math 26, 11 - 21 (2004).

[Mu] Mumford, D., *Lectures on curves on an algebraic surface*, Annals of Math Studies 59, Princeton Univ. Press, princeton (1966).

[RS] Rotthaus, C. and Sega, L.M., *Some properties of graded local cohomology modules*, J. Algebra 283, 232 - 247 (2005).

[Z] Zariski, O., *The theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface*, Ann. Math 76 (1962), 560 – 615.

Steven Dale Cutkosky, Department of Mathematics, University of Missouri, Columbia, MO 65211, USA

E-mail address: `CutkoskyS@missouri.edu`

Jürgen Herzog, Fachbereich Mathematik und Informatik, Universität Duisburg-Essen, Campus Essen, 45117 Essen, Germany

E-mail address: `juergen.herzog@uni-essen.de`